First semestral backpaper exam 2013
B.Math. (Hons.) IInd year
Algebra III — B.Sury
Answer any FIVE questions.

Q 1. State and prove the Chinese remainder theorem for two ideals in a commutative ring.

Q 2. If A is a commutative ring with unity, prove that an element $f \in A[X]$ which is in the intersection of all maximal ideals must be a nilpotent element.

Q 3. Let I, J, K be ideals of a commutative ring A with unity. If P is a prime ideal of A containing the product IJK, then show that P contains I, J or K.

Q 4. Show that $\mathbf{Z}[X]$ is not a Euclidean domain.

Q 5. Let M be a free module of finite rank over a PID. Show that any submodule $N \neq 0$ of M is free, of rank $r \leq rank(M)$.

Q 6. Use the fact that $\mathbf{Z}[i]$ is a UFD to determine all solutions of $x^2+1 = y^3$ in integers x, y.

Q 7. Given a matrix $M \in M_n(K)$, where K is a field, what is meant by its rational canonical form? Further, by assuming the existence of the rational canonical form, compute the characteristic polynomial of M.

Q 8. Prove that the polynomial $X^{100} - 123123X^{28} + 110$ cannot take the values ± 33 over integers.